Chapter 6

THE BINOMIAL THEOREM

In Chapter 1 we defined $\binom{n}{r}$ as the coefficient of $a^{n-r}b^r$ in the expansion of $(a + b)^n$, and tabulated these coefficients in the arrangement of the Pascal Triangle:

n	Coefficients of $(a + b)^n$												
0							1						
1						1		1					
2					1		2		1				
3				1		3		3		1			
4			1		4		6		4		1		
5		1		5		10		10		5		1	
6	1		6		15		20		15		6		1
...	.	.	.	.	.	.	.	.					

We then observed that this array is bordered with 1's; that is, $\binom{n}{0} = 1$ and $\binom{n}{n} = 1$ for $n =$ 0, 1, 2, ... We also noted that each number inside the border of 1's is the sum of the two closest numbers on the previous line. This property may be expressed in the form

(1) $$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.$$

This formula provides an efficient method of generating successive lines of the Pascal Triangle, but the method is not the best one if we want only the value of a single binomial coefficient for a large n, such as $\binom{100}{3}$. We therefore seek a more direct approach.

It is clear that the binomial coefficients in a diagonal adjacent to a diagonal of 1's are the numbers 1, 2, 3, ... ; that is, $\binom{n}{1} = n$. Now let us consider the ratios of binomial coefficients to the previous ones on the same row. For $n = 4$, these ratios are:

(2) $$4/1,\ 6/4 = 3/2,\ 4/6 = 2/3,\ 1/4.$$

For $n = 5$, they are

(3) $$5/1,\ 10/5 = 2,\ 10/10 = 1,\ 5/10 = 1/2,\ 1/5.$$

The ratios in (3) have the same pattern as those in (2) if they are rewritten as

$$5/1,\ 4/2,\ 3/3,\ 2/4,\ 1/5.$$

It is easily seen that this pattern also holds on the line for $n = 8$, and that the coefficients on that line are therefore:

(4) $$1,\ \frac{8}{1},\ \frac{8}{1}\cdot\frac{7}{2},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3},\ \frac{8}{1}\cdot\frac{7}{2}\cdot\frac{6}{3}\cdot\frac{5}{4},\ \ldots .$$

The binomial coefficient $\binom{8}{3} = \frac{8\cdot7\cdot6}{1\cdot2\cdot3}$ can be rewritten as

$$\frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{1\cdot2\cdot3\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{8!}{3!5!}.$$

Similarly, $\binom{8}{2} = \frac{8!}{2!6!}$ and $\binom{8}{4} = \frac{8!}{4!4!}$.

This leads us to conjecture that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ holds in all cases. We prove this by mathematical induction in the following theorem.

THEOREM: If n and r are integers with $0 \le r \le n$, then

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof: If $n = 0$, the only allowable value of r is 0 and $\binom{0}{0} = 1$. Since

$$\frac{n!}{r!(n-r)!} = \frac{0!}{0!0!} = 1$$

the formula holds for $n = 0$.

Now let us assume that it holds for $n = k$. Then

$$\binom{k}{r-1} = \frac{k!}{(r-1)!(k-r+1)!}, \binom{k}{r} = \frac{k!}{r!(k-r)!}.$$

Using (1), above, we now have

$$\begin{aligned}
\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r} &= \frac{k!}{(r-1)!(k-r+1)!} + \frac{k!}{r!(k-r)!} \\
&= \frac{k!r}{(r-1)!r(k-r+1)!} + \frac{k!(k-r+1)}{r!(k-r)!(k-r+1)} \\
&= \frac{k!(r+k-r+1)}{r!(k-r+1)!} \\
&= \frac{k!(k+1)}{r!(k-r+1)!} \\
&= \frac{(k+1)!}{r!(k-r+1)!}.
\end{aligned}$$

Since the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

is the theorem for $n = k + 1$, the formula is proved for all integers $n \geq 0$, with the exception that our proof tacitly assumes that r is neither 0 nor $k + 1$; that is, it deals only with the coefficients inside the border of 1's. But the formula

$$\binom{k+1}{r} = \frac{(k+1)!}{r!(k-r+1)!}$$

shows that each of $\binom{k+1}{0}$ and $\binom{k+1}{k+1}$ is $\frac{(k+1)!}{0!(k+1)!} = 1.$

Hence the theorem holds in all cases.

The above theorem tells us that the coefficient of $x^r y^s$ in $(x+y)^n$ is

$$\frac{n!}{r!s!}.$$

Since this expression has the same value when r and s are interchanged, we again see that the binomial coefficients have the symmetry relation

$$\binom{n}{r} = \binom{n}{n-r}.$$

By writing out the factorials more explicitly, we see that

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$= \frac{n(n-1)(n-2)...(n-r+1)(n-r)(n-r-1)...2\cdot 1}{1\cdot 2\cdot 3...r(n-r)(n-r-1)...2\cdot 1}.$$

Cancelling common factors, we now have

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ ...\ (n-r+1)}{1\cdot 2\cdot 3\ ...\ r}.$$

This is the alternate form of the theorem illustrated for $n = 8$ in (4), above.

We can now rewrite the expansion of $(a+b)^n$ in the form

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + ...$$

$$+\frac{n(n-1)...(n-r+1)}{1\cdot 2\ ...\ r}a^{n-r}b^r + \cdots + b^n.$$

This last formula it generally called the **Binomial** **Theorem.**

The formulas

$$\binom{n}{0} = 1, \binom{n}{r} = \frac{n(n-1)...(n-r+1)}{1 \cdot 2 \ ... \ r} \text{ for } r > 0$$

enable us to extend the definition of $\binom{n}{r}$, previously defined only for integers n and r with $0 \le r \le n$, to allow n to be any integer. We then have, for example, $\binom{2}{5} = 0$, $\binom{-2}{7} = -8$, and $\binom{-3}{8} = 45$.

It can easily be shown that the formula

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

holds with the extended definition as it did with the original definition.

Now the identity

$$2\binom{m}{2} + \binom{m}{1} = 2\frac{m(m-1)}{2} + m = m^2 - m + m = m^2$$

holds for all integers m, and we can use the formulas

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} + ... + \binom{n}{1} = \binom{n+1}{2}$$

$$\binom{1}{2} + \binom{2}{2} + \binom{3}{2} + ... + \binom{n}{2} = \binom{n+1}{3}$$

to show that

$$
\begin{aligned}
&1^2 + 2^2 + \dots + n^2 \\
&\quad = \left[2\binom{1}{2} + \binom{1}{1}\right] + \left[2\binom{2}{2} + \binom{2}{1}\right] + \dots + \left[2\binom{n}{2} + \binom{n}{1}\right] \\
&\quad = 2\left[\binom{1}{2} + \binom{2}{2} + \dots + \binom{n}{2}\right] + \left[\binom{1}{1} + \binom{2}{1} + \dots + \binom{n}{1}\right] \\
&\quad = 2\binom{n+1}{3} + \binom{n+1}{2} \\
&\quad = 2\frac{(n+1)n(n-1)}{6} + \frac{(n+1)n}{2} \\
&\quad = \frac{2n^3 - 2n}{6} + \frac{3n^2 + 3n}{6} \\
&\quad = \frac{2n^3 + 3n^2 + n}{6} \\
&\quad = \frac{n(n+1)(2n+1)}{6}.
\end{aligned}
$$

Frequently in mathematical literature a short notation for sums is used which involves the Greek capital letter sigma, written $\sum$. In this notation,

$$a_1 + a_2 + \dots + a_n$$

is written as

$$\sum_{i=1}^{n} a_i$$

and the auxiliary variable i is called the **index of summation**. Thus, for example,

$$
\begin{aligned}
\sum_{i=1}^{5} i &= 1 + 2 + 3 + 4 + 5 = 15 \\
\sum_{i=1}^{6} 1 &= 1 + 1 + 1 + 1 + 1 + 1 = 6 \\
\sum_{j=1}^{n-1} j^2 &= 1^2 + 2^2 + 3^2 + \dots + (n-1)^2.
\end{aligned}
$$

Under the capital sigma, one indicates the symbol that is used as the index of summation and the initial value of this index. Above the sigma, one indicates the final value. The general polynomial $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$ of degree n can be written as

$$\sum_{k=0}^{n} a_k x^{n-k}.$$

One easily sees that

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$$

since

$$\begin{aligned}\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i &= (a_1 + a_2 + \dots + a_n) + (b_1 + b_2 + \dots + b_n)\\ &= (a_1 + b_1) + (a_2 + b_2) + \dots + (a_n + b_n)\\ &= \sum_{i=1}^{n} (a_i + b_i).\end{aligned}$$

Also, $\sum_{i=1}^{n} (ca_i) = c\sum_{i=1}^{n} a_i$, the proof of which is left to the reader. However,

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \neq \sum_{i=1}^{n} (a_i b_i)$$

as can easily be shown by counterexample. (See Problem 19 of this chapter.)

A corresponding notation for products uses the Greek letter pi:

$$\prod_{i=1}^{n} a_i = a_1 a_2 \dots a_n.$$

In this notation, $n!$ for $n \geq 1$ can be expressed as $\prod_{k=1}^{n} k$.

In solving problems stated in terms of the sigma or pi notation, it is sometimes helpful to rewrite the expression in the original notation.

Problems for Chapter 6

1. Find each of the following:

   (a) The coefficient of x^4y^{16} in $(x + y)^{20}$.
   (b) The coefficient of x^5 in $(1 + x)^{15}$.
   (c) The coefficient of x^3y^{11} in $(2x - y)^{14}$.

2. Find each of the following:

   (a) The coefficient of $a^{13}b^4$ in $(a + b)^{17}$.
   (b) The coefficient of a^{11} in $(a - 1)^{16}$.
   (c) The coefficient of a^6b^6 in $(a - 3b)^{12}$.

3. Find integers a, b, and c such that $6\binom{n}{3} = n^3 + an^2 + bn + c$ for all integers n.

4. Find integers p, q, r, and s such that $4!\binom{n}{4} = n^4 + pn^3 + qn^2 + rn + s$ for all integers n.

5. Prove that $\binom{n}{3} = 0$ for $n = 0, 1, 2$.

6. Given that k is a positive integer, prove that $\binom{n}{k} = 0$ for $n = 0, 1, \ldots, k-1$.

7. Find $\binom{-1}{r}$ for $r = 0, 1, 2, 3, 4,$ and $5.$.

8. Find $\binom{-2}{r}$ for $r = 0, 1, 2, 3, 4,$ and $5.$.

9. Prove that $\binom{-3}{r} = (-1)^r\binom{r+2}{2}$ for $r = 0, 1, 2, \ldots$.

10. Prove that $\binom{-4}{r} = (-1)^r \binom{r+3}{3}$ for $r = 0, 1, 2, \ldots$.

11. Let m be a positive integer and r a non-negative integer. Express $\binom{-m}{r}$ in terms of a binomial coefficient $\binom{n}{k}$ with $0 \le k \le n$.

12. In the original definition of $\binom{n}{r}$ as a binomial coefficient, it was clear that it was always an integer. Explain why this is still true in the extended definition.

13. Show that $\binom{n}{a}\binom{n-a}{b} = \dfrac{n!}{a!b!(n - a - b)!}$ for integers a, b, and n, with $a \ge 0$, $b \ge 0$, and $n \ge a + b$.

14. Given that $n = a + b + c + d$ and that a, b, c, and d are non-negative integers, show that

$$\binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}\binom{n-a-b-c}{d} = \frac{n!}{a!b!c!d!}.$$

15. Express $\sum_{k=1}^{n} [a + (k - 1)d]$ as a polynomial in n.

16. Express $\prod_{k=1}^{n} (2k)$ compactly without using the Π notation.

17. Show that $\prod_{k=1}^{n} a_k = \prod_{j=0}^{n-1} a_{j+1}$.

18. Show that $\sum_{k=1}^{n-2} b_k = \sum_{i=3}^{n} b_{i-2}$.

19. Evaluate $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} b_i\right)$ and $\sum_{i=1}^{2} (a_i b_i)$ and show that they are not always equal.

20. Show that $\left(\prod_{i=1}^{n} a_i\right)\left(\prod_{i=1}^{n} b_i\right) = \prod_{i=1}^{n} (a_i b_i)$.

21. Prove by mathematical induction that $\sum_{i=0}^{n} \binom{s+i}{s} = \binom{s+1+n}{s+1}$.

22. Prove that $\sum_{j=0}^{n} \binom{s+j}{j} = \binom{s+1+n}{n}$.

23. Express $\sum_{k=1}^{n-2} \frac{k(k+1)}{2}$ as a polynomial in n.

24. Express $\sum_{k=1}^{n-2} \binom{k+1}{k-1}$ as a polynomial in n.

25. Write $6\left[\binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n, and then use the fact that $\binom{n}{r}$ is always an integer to give a new proof that $n(n^2 + 5)$ is an integral multiple of 6 for all integers n.

26. (a) Write $4!\left[\binom{n}{4} + \binom{n}{3} + \binom{n}{2} + \binom{n}{1}\right]$ as a polynomial in n.

(b) Show that $n^4 - 2n^3 + 11n^2 + 14n$ is an integral multiple of 24 for all integers n.

27. Find numbers s and t such that $n^3 = n(n - 1)(n - 2) + sn(n - 1) + tn$ holds for $n = 1$ and $n = 2$.

28. Find numbers a and b such that $n^3 = 6\binom{n}{3} + a\binom{n}{2} + b\binom{n}{1}$ for all integers n.

29. Find numbers r, s, and t such that $n^4 = n(n - 1)(n - 2)(n - 3) + rn(n - 1)(n - 2) + sn(n - 1) + tn$ for $n = 1, 2$, and 3. Using these values of r, s, and t, show that

$$n^4 = 24\binom{n}{4} + 6r\binom{n}{3} + 2s\binom{n}{2} + t\binom{n}{1}$$

for all integers n.

30. Find numbers a, b, c, and d such that

$$n^5 = 5!\binom{n}{5} + a\binom{n}{4} + b\binom{n}{3} + c\binom{n}{2} + d\binom{n}{1}.$$

31. Express $\sum_{k=1}^{n} k^4$ as a polynomial in n.

32. Express $\sum_{k=1}^{n} k^5$ as a polynomial in n.

33. We define a sequence $S_0, S_1, S_2, \ldots$ as follows: When n is an even integer $2t$, let

$S_n = S_{2t} = \sum_{j=0}^{t}\binom{t+j}{t-j}$. When n is an odd integer $2t + 1$, let $S_n = S_{2t+1} = \sum_{j=0}^{t}\binom{t+1+j}{t-j}$.

Prove that $S_{2t} + S_{2t+1} = S_{2t+2}$ and $S_{2t+1} + S_{2t+2} = S_{2t+3}$ for $t = 0, 1, 2, \ldots$.

34. For the sequence defined in Problem 33, prove that S_n is the Fibonacci number F_{n+1}.

*35. Prove the following property of the Fibonacci numbers:

$$\sum_{j=0}^{n}\binom{n}{j}(-1)^j F_{s+2n-2j} = F_{s+n}.$$

*36 Prove an analogue of the formula of Problem 35 for the Lucas numbers.

37 Find a compact expression, without using the sigma notation, for

$$1\cdot n + 2(n-1) + 3(n-2) + \cdots + (n-1)\cdot 2 + n\cdot 1,$$

that is, for $\sum_{k=0}^{n-1}(k+1)(n-k)$.